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The Effect of External Noise in the Lorenz Model of the Bénard Problem

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The effect of external random forces on the static and dynamic behavior of the Lorenz model is investigated. Results of a numerical calculation in the conductive, convective, and turbulent regimes are reported. The properties of static and time-dependent correlation functions of the three degrees of freedom of the model are analyzed for varying strength of the external noise level and compared with the behavior of the unforced system.

KEY WORDS: Turbulence; Lorenz model; random behavior of nonlinear differential equations; response to external random forces; dynamic correlation and response functions.

1. INTRODUCTION

In the last few years an increasing number of physicists have directed their research activities toward investigating the complex behavior of nonlinear model systems described by seemingly simple deterministic differential or difference equations. The best-known examples of such models—undergoing a transition to chaotic states upon variation of a parameter—are presumably the Lorenz model[(1)] of the Rayleigh Bénard problem and models of seasonally breeding insect populations without generation overlap.[(2)] However, there is a wealth of similar systems of nonlinear differential equations[3] and difference equations[(7,8)] whose chaotic state is charac-

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[3] References 3–5. Other systems, like the laser and certain chemical reactions, show transitions to stochastic behavior as well. For an introduction see Ref. 6.

terized by apparently random-looking motion on attractors of complicated structure.[9]

The trajectories in the phase space of these systems and the bifurcation sequence leading to the erratic motion have been studied intensively in computer calculations.[3–5,7,10–13] Liapunov characteristic exponents[11–15] and Poincaré maps, symbolic transition dynamics, and other mathematical machinery[8,16–18] have been used. Possible universal properties of the bifurcation sequences have been studied using scaling approaches.[13,19]

The somewhat intriguing possibility of doing statistical mechanics for these systems beyond the calculations performed by McLaughlin[20] was pointed out in the fundamental paper of Ref. 3. That prompted one of us to make an analytical and numerical investigation[21] of the statistical dynamics of the Lorenz model in the turbulent regime. This approach has since been continued[22] and complemented with other techniques.[23] Aizawa and Shimada[24] performed a numerical calculation in order to elucidate the ergodicity of motion on the Lorenz attractor. Unfortunately, the precise numerical procedure leading to their results were not given. Nakamura[5] made a numerical analysis of transition probabilities to describe the spreading of trajectories over the phase space of the Lorenz system and of a 40-dimensional model which simulates the Gunn instability. Statistical aspects of the chaotic behavior of one-dimensional nonlinear difference equations have also been studied recently.[25] It is an interesting question how these stochastic systems respond to external random forces. A detailed answer in the case of a numerically simple system like the Lorenz model might shed some light on the old problem of statistical turbulence theory: What is the precise relation between external (random) forces necessary to sustain (statistically stationary) turbulence and the statistics of the turbulent flow? Limit cycle systems which show a transition to random behavior under external periodic forcing have been studied quite intensively.[26] Also, transition mechanisms which are induced by external noise have been explored lately.[27] On the other hand, there are only few investigations of systems which like the Lorenz model, display transitions to chaotic states also in the absence of external forces: Nakamura[5] applied small random forces with a uniform distribution over a finite interval to the Lorenz system and his 40-mode model. He compared the projection of trajectories onto a two-dimensional surface with and without external noise and found evidence for a threshold behavior—in the presence of large enough forces, trajectories escaped an attractor, wandering to another one. McLaughlin[20] studied possible modifications of the point attractors and of the strange attractor of the Lorenz model caused by external noise coupled to one mode. He also worked out the direct-interaction-approximation[28] for this system and compared its results with computer calculations done for the correlation functions of two of the three Lorenz model modes.

In this paper we discuss the results of a numerical investigation of the effect of external noise on the behavior of the Lorenz model in the conduction, convection, and turbulent regime.[4] We calculated static as well as dynamical correlations between the three degrees of freedom of the model and compared with the behavior of the unforced system.[1,3,21] In Section 2 we display the model equations and some numerical details related to the calculation of trajectories and averages in the presence of random forces. In Section 3 we present and discuss our results obtained in the three regimes cited above. The last section gives a summary.

2. THE MODEL

The subject of our investigation is the Lorenz model[1] with random forces, defined by the equations of motion[5]

$$x = \sigma y + f_x \tag{2.1a}$$

$$y = (R-1)x - (\sigma+1)y - xz + f_y \tag{2.1b}$$

$$z = -bz + xy + x^2 + f_z \tag{2.1c}$$

x, y, and z are dimensionless variables and $f_i(t)$ $(i = x, y, z)$ a white noise source, chosen according to a probability distribution $P\{f\}$. For the main part of the paper we will use Gaussian distributed forces with zero mean, characterized by their second moment

$$\langle f_i(t) f_j(t') \rangle = \delta_{ij} D \delta(t-t') \tag{2.2}$$

Uniformly distributed forces will be discussed briefly later on. The parameters $\sigma = 10$ and $b = 8/3$ are held fixed at their standard values.[1,29] Depending on the Rayleigh number R, the solutions of (2.1) without random forces show qualitatively different behavior. In the conduction range $R < 1$ the trivial steady-state solution $x = y = z = 0$ is stable, while in the convection regime $R > 1$ two other steady-state solutions

$$\pm x_0 = \pm\left[b(R-1)\right]^{1/2}, \qquad y_0 = 0, \qquad z_0 = R - 1 \tag{2.3}$$

become stable and remain so up to $R = R_T = \sigma(\sigma + b + 3)/(\sigma - b - 1) = 24.74$. However, there are "preturbulent" states[18] below R_T due to

[4] For a physical interpretation of the Lorenz model variables and their relation to the original Bénard system see, e.g., Ref. 3.

[5] We have substituted for the variable y of the original Lorenz equations $y - x$ to arrive at (2.1).

chaotic orbits. For $R > R_T$ no stable steady-state solution exists. The trajectories are nonperiodic and irregular for a certain range of R values above R_T, while for still larger R periodic motions are found to exist[(12,13,15,22)] for special ranges of R.

Our main interests are the statistical properties of (2.1) as displayed in correlation functions in the presence of random forces, whose strength D is allowed to vary. The basic quantity is the matrix of correlation functions

$$C_{ij}(t) = \langle A_i(t) A_j(0) \rangle \tag{2.4}$$

which depend only on time differences because of the time translational invariance of (2.2). Here $A_i(t)$ is the fluctuation of one of the dynamical variables x, y, z and $\langle \cdots \rangle$ denotes an average over random forces. Assuming that the system is ergodic, we replace all averages of functions F of the variables x, y, z over random forces by time averages.[6]

In the convection regime $1 < R < R_T$ a trajectory of the unforced Lorenz model gets attracted toward (x_0, y_0, z_0) or toward $(-x_0, y_0, z_0)$, depending on whether the starting point was in the basin of attraction $x < 0$ of the first attractor or in the basin of attraction $x > 0$ of the second one. Also, for some large R values the unforced Lorenz model has symmetry-broken solutions. Pairs of asymmetric periodic attractors have been reported[(12,15,22)] which map into each other under the symmetry transformation $(x, y, z) \to (-x, -y, z)$ and different symmetry-violating limit cycles were shown[(15,22)] to attract different sets of initial points. The correspondence between initial condition and final state attractor in the two described cases is, of course, not surprising, given that we are dealing with first-order differential equations. It seems, however, worth mentioning that some of the relations[(21)] between equal-time averages following from the symmetry of the unforced Lorenz equations have been found[(22)] to be weakly violated in the turbulent regime. This implies that equal-time correlations show, even in the chaotic range, some memory of the initial conditions over those times covered by the computer calculations.[(22)]

In the forced system (2.1), however, we did not find numerical evidence for a violation of the symmetry $(x, y, z, f_x, f_y, f_z) \to (-x, -y, z, -f_x, -f_y, f_z)$. The time average of x in the convection range $1 < R < R_T$ was zero and there was no indication of symmetry violations in other equal-time correlations either. Some calculations done with different starting points support the conclusion that time averages are independent of trajectories in the presence of Gaussian random forces. The latter seem to destroy any memory of the starting point. However, we did not investigate that problem systematically.

[6] For a discussion of the reliability of time averages in numerical studies of stochastic behavior see Ref. 30.

The forward difference method was used to integrate the system of differential equations (2.1). The time step was $\Delta t = 10^{-3}$ and a total averaging time of $T = 1700$ ensured numerical convergence of time averages. The random force $f(t)$ was approximately taken to be a random constant $\hat{f}_i/\Delta t$ for $t_i \leqslant t \leqslant t_i + \Delta t$, so that $D = \langle \hat{f}_i \hat{f}_i \rangle / \Delta t$. We thus generated for the three forces in (2.1) some 10^6 random numbers with two random number generators coupled together in such a way as to destroy possible machine-inherent correlations. To test the reliability of our procedures we calculated numerically correlations of the linearized forced equations (2.1) and compared the results with the analytic expressions given later. The agreement was very good, mostly within pencil's width.

3. NUMERICAL RESULTS

3.1. Conduction Regime: $R < 1$

In Fig. 1 we show some of the results of the numerical integration of Eqs. (2.1) for $R = 1/2$ and a Gaussian forcing spectrum with $D = 1/4$, 1, and 4. The experimental curves are compared to the solution of (2.1) linearized around the steady state $x = y = z = 0$ of the unforced system. Neglecting nonlinear terms, (2.1) can easily be solved for the correlation functions. For example,

$$C_{xx}^{0}(\omega) = D \frac{\omega^2 + (\sigma + 1)^2 + \sigma^2}{\left[\omega^2 - \sigma(R-1)\right]^2 + \omega^2(\sigma+1)^2} \tag{3.1}$$

has two pairs of complex conjugate poles, where Fourier transforms have

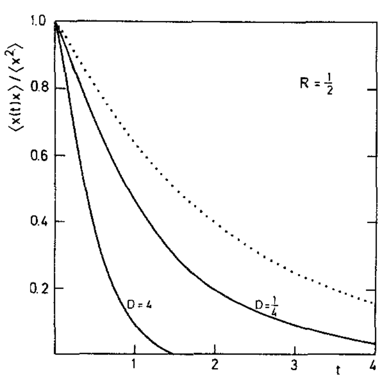

Fig. 1. Normalized correlation function $\langle x(t)x \rangle / \langle x^2 \rangle$ in the conductive regime for $D = 1/4$ and 4, compared to the solution of the linearized equations (· · ·).

been introduced according to

$$C_{ij}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} C_{ij}(t) \tag{3.2}$$

The corresponding time-dependent correlation function $C_{xx}^0(t)$ is the sum of two exponentials with inverse relaxation times which are independent of the strength D of the fluctuating force. D enters the linearized correlation functions only via an overall factor, so that the normalized quantity $\tilde{C}_{xx}^0(t) = C_{xx}^0(t)/C_{xx}^0\ (t=0)$ is independent of D.

This function is shown in Fig. 1 by dots. Clearly the actual relaxation times of the nonlinear system (2.1) do depend on the strength of the fluctuating force. The autocorrelation of x gets damped out faster with increasing noise level D. The correlation functions $C_{yy}(t)$ and $C_{zz}(t)$ show roughly the same behavior. For all values of D considered here the diagonal elements of the correlation matrix (2.4) are monotonically decreasing functions of time. The relaxation times show the overall tendency to decrease with increasing forcing strength, the effect being most pronounced for $C_{xx}(t)$.

The numerical results for equal-time correlation functions show that fluctuations increase with increasing noise level D. Whereas the linearized theory predicts the expectation value of $\langle z \rangle$ to be strictly zero, the full equations (2.1) lead to a finite $\langle z \rangle$, increasing with increasing D. Furthermore, the linear theory overestimates the fluctuations in x and underestimates those in z, the discrepancies being largest for large D. We conclude that the linear theory is inadequate for random forces with strength D sufficiently large to frequently kick trajectories out into phase space regions far away from the former steady-state point at the origin. Then the nonlinear terms in (2.1) become important, producing erratic motion as in the turbulent regime $R > R_T$ of the unforced system. Its phase mixing properties lead to decorrelation beyond that caused by external noise. The above argumentation is supported also by inspecting the typical behavior of trajectories in the presence of random forces. For most of the time it resembles Brownian motion in the phase space restricted to the surrounding of the origin. When kicked out far enough, however, trajectories spiral around in much the same way as they do in the unforced system beyond R_T before returning close to the origin. But even for the smallest forcing we found the average $\langle x^2 \rangle$ of one of the nonlinear terms entering (2.1c) to be larger in size than the average linear term $b\langle z \rangle$.

3.2. Convective Regime: $1 < R < R_T$

In the absence of random forces, the system is attracted in general[18] to one of the stable steady states $(\pm x_0, y_0, z_0)$ depending on its initial state. If random forces are applied, the trajectories are no longer confined to one of the steady-state points. In particular, motion from one point to the other becomes possible.

We performed three numerical calculations at $R = 10$ for a Gaussian forcing spectrum with $D = 4$, 16, and 64. In all cases we found $\langle x \rangle = 0$. This result shows that crossings from one basin of attraction to the other do occur for Gaussian-distributed random forces. Simple equal-time correlations were found to be comparable in magnitude to the corresponding steady-state values and not to depend significantly on D, except for $\langle y^2 \rangle$, which vanishes for $D = 0$.

The time-dependent correlation function $\tilde{C}_{xx}(t)$ is, for all values of D shown in Fig. 2a, an almost monotonic function of time. However, the decay time of correlations increases drastically for decreasing D. In the limit $D \to 0$ it takes longer and longer for a point in phase space to escape from one of the basins of attraction to move to the other one. As a consequence correlations, i.e., memory of the initial state, persist over an extended time scale. This phenomenon is analogous to critical slowing down in a second-order phase transition, if we interpret the state $D = 0$ with the two stable solutions $x = \pm x_0$ as a state of broken symmetry. Indeed, the system (2.1) can be rewritten[12,31] into an integrodifferential equation for $x(t)$ alone, which describes the motion of a particle in a double-well potential in the presence of friction, and a force[31] depending on time and the history of $x(t)$. For $1 < R < R_T$ the above force is too small to overcome the central barrier of the well and the particle gets trapped in one of the minima at $\pm x_0$ unless there are external (random) forces strong enough to kick it into the other well. The slow decay of correlations in $C_{xx}(t)$ obliterates the accuracy of time averages (2.5) evalu-

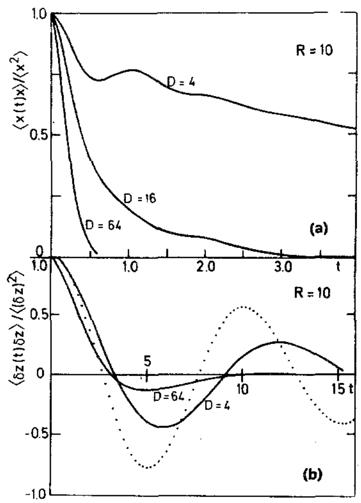

Fig. 2. Normalized correlation functions in the convective regime for various noise levels D, compared to the solution of the linearized equations (· · ·).

ated with finite T. Our data for $R = 10$ and $D = 4$ are therefore less reliable than the other results presented in this paper.

Figure 2b shows, as an example which is also representative for $\tilde{C}_{yy}$, the normalized correlations $\tilde{C}_{zz}(t)$ for $D = 4$ and 64. They display damped oscillations with roughly the period of the linearized theory discussed below. The damping, however, increases drastically with growing noise level D. Also shown are the correlations following straightforwardly from the linearized version of Eqs. (2.1) around one of the point attractors $(\pm x_0, y_0, z_0)$. Within the above linear theory the normalized correlation functions $\tilde{C}_{ii}(t) = C_{ii}(t)/C_{ii}(0)$ are again independent of D. Their time dependence is determined by three poles in the complex frequency plane.[3] One is purely imaginary, the other two have finite real parts of opposite sign, reflecting spiraling motion around one of the attractors $(\pm x_0, y_0, z_0)$. This gives rise to the oscillatory behavior of $\tilde{C}_{yy}(t)$ and $\tilde{C}_{zz}(t)$ in Fig. 2b. These two correlation functions show a qualitative agreement between the D-independent linearized theory and "experiments" only for small D. In that limit the agreement is best for short times.

However, the linear theory for $\tilde{C}_{xx}(t)$ is qualitatively wrong. Depending only on x_0^2, it does not properly reflect the fact that there are really two different attractors at $x = \pm x_0$ between which the orbit can move back and forth in the presence of random forces. If the noise level is high, the trajectory frequently commutes between the two basins $x \gtrless 0$ of attraction of the linearized theory. That is the only mechanism to effectively destroy the memory of the basin, $x < 0$ or $x > 0$, in which the trajectory started. According to Fig. 2a, a high noise level seems to be realized for $D = 16$. On the other hand, if D is sufficiently small, large enough forces to kick the orbit into the other basin are rare. The trajectory therefore stays a long time in the same basin of attraction close to the attractor (small forcing!) where the motion is dominated by the characteristic frequencies of the system linearized around $(\pm x_0, y_0, z_0)$. Hence one expects oscillatory behavior for $\tilde{C}_{xx}(t)$ superimposed upon the slow decay of correlations caused by the rare crossings. Such a situation seems to prevail for $D = 4$ in Fig. 2a.

3.3. Turbulent Regime: $R > R_T$

The Lorenz model without random forces shows a transition to chaotic behavior at $R = R_T^{(1)}$. Above the threshold value R_T trajectories are nonperiodic and irregular, while for still larger values of R, periodicity strips were found to exist.[12,13,15,22] The nonperiodic solutions belong to a strange attractor type of solution; their statistical properties have been discussed by one of us.[21] To compare with these results we performed a numerical calculation for $R = 30$ and chose D to be small compared with the average

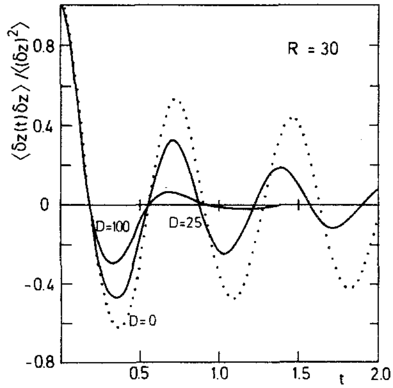

Fig. 3. Normalized correlation function $\langle \delta z(t)\, \delta z\rangle / \langle \delta z^2 \rangle$ in the turbulent regime for various noise levels D.

velocity squares $\langle \dot{x}_i^2 \rangle$ of the unforced system. Our results for $D = 25$ and 100 are presented in Figs. 3 and 4.

In Table I we list some of the experimental results for equal-time correlations. All fluctuations are increased in the presence of random forces. The changes in $\langle z \rangle$ and in the fluctuations $\langle (\delta z)^2 \rangle$ and $\langle x^2 \rangle$, however, are rather small, while the fluctuations of y are enhanced considerably in the presence of random forces. Another effect of the forcing is that the correlation $\langle xy \rangle$ which vanishes[21] for $D = 0$ is finite in the presence of external noise and equals $-\langle x f_x \rangle / \sigma$. Moreover, $\langle x(t) y \rangle$ shows no definite symmetry under time inversion.

For $D = 0$ and $R = 30$ the motion is nonperiodic, and the trajectories spiral back and forth around the former attractors $(\pm x_0, y_0 z_0)$. Also, when externally forced, the system's averaged quantities $\langle z \rangle$ and $\langle x^2 \rangle$ stay close to z_0 and x_0^2, respectively, as shown in Table I.

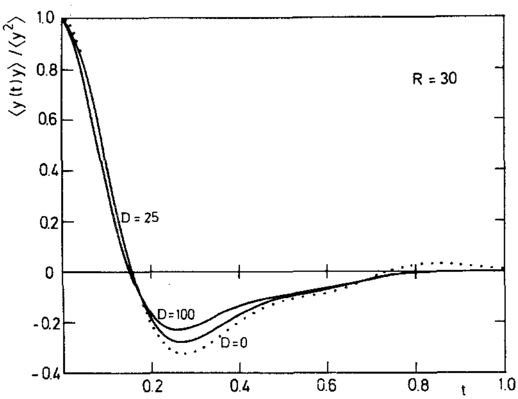

Fig. 4. Same as Fig. 3 for $\langle y(t) y\rangle / \langle y^2 \rangle$.

Table I. Equal-Time Correlations in the Turbulent Regime for Various Noise Levels D

$R = 30$	$D = 0$	$D = 25$	$D = 100$	
$\langle z \rangle$	25.6	27.8	28.2	$z_0 = 29$
$\langle x^2 \rangle$	68.1	79.2	81.1	$x_0^2 = 77.33$
$\langle y^2 \rangle$	21.1	31.6	36.6	$y_0^2 = 0$
$\langle z^2 \rangle$	734	854	880	$z_0^2 = 841$
$\langle (\delta z)^2 \rangle$	78.6	81.2	84.8	—
$\langle (\delta z)^2 \rangle^{1/2} / \langle z \rangle$	0.35	0.32	0.33	—

In Fig. 3 we show the normalized time-dependent correlation function $\tilde{C}_{zz}(t) = \langle \delta z(t) \delta z \rangle / \langle (\delta z)^2 \rangle$. For $D = 0$ this function displays exponentially damped oscillations with frequency[21] $\Omega_z = (\langle \dot{z}^2 \rangle / \langle (\delta z)^2 \rangle)^{1/2}$. The total decay time is of the order of 10. For small times the fluctuating forces considered here have little effect, while for larger times they enhance the decay of correlations. The period of oscillation is only slightly shifted with respect to $D = 0$, whereas the total decay time is considerably decreased—being comparable to the period of oscillation for $D = 100$.

The normalized time-dependent correlation function $\tilde{C}_{yy}(t) = \langle y(t) y \rangle / \langle y^2 \rangle$ is plotted in Fig. 4. Independent of D, this function shows a sharp initial decay in a time of order 0.2. Then, within a time of order 1 the system completely loses its memory of the initial value $y(0)$ while undergoing on the average a few oscillations around the origin. Since this decay is so fast already without external noise, the addition of random forces has less effect on dynamic correlations of y than on those of z, whose relaxation time is much larger for $D = 0$. This confirms our observation that the external noise applied here has little effect on the behavior of normalized correlation functions over times $t \ll 1$.

For $R = 30$ we performed a calculation with a constant distribution of random numbers having the same second moment as the Gaussian distribution: $D = 100$. With this choice, differences in correlation functions for a constant and a Gaussian distribution lie within the error bars.

It is instructive to compare dynamical correlations in the two different regimes $1 < R < R_T$ and $R > R_T$. To do so we define a reduced time $\tilde{t} = tR^{1/2}$. This choice is motivated by the observation that for $D = 0$ the characteristic frequency squares[7] $\Omega_i^2 = \langle \dot{A}_i^2 \rangle / \langle A_i^2 \rangle$ scale like R,[21–23] and

[7] The graph of the frequency $\Omega_\infty = (\langle \dot{y}^2 \rangle / \langle y^2 \rangle)^{1/2}$ in Fig. 6 of Ref. 21 is wrong. Instead of

$$\Omega_\infty = \left\{ \langle x^2 [z^2 - (R-1)^2] \rangle / \langle y^2 \rangle - 2\sigma(R-1) - (\sigma+1)^2 \right\}^{1/2}$$

[Eq. (38b)], the graph represents

$$\left\{ \langle x^2 [z^2 - (R-1)^2] \rangle / \langle y^2 \rangle - 2(R-1) - (\sigma+1)^2 \right\}^{1/2}$$

which has to be shifted downward to obtain Ω_∞. One of us (ML) would like to thank B. Sonneborn-Schmick and S. Grossmann for pointing out this error.

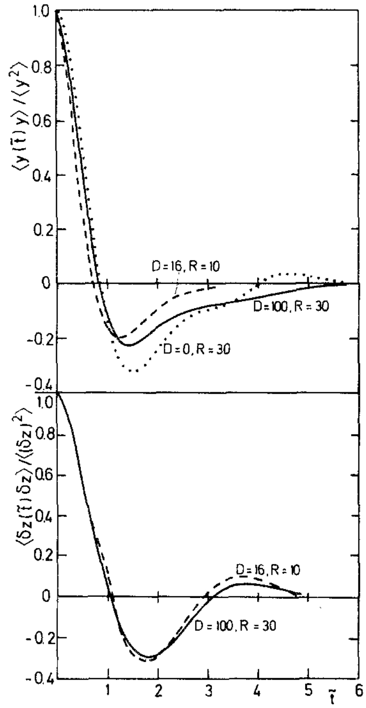

Fig. 5. Normalized correlation functions in the convective and turbulent regimes as a function of reduced time $\tilde{t} = tR^{1/2}$.

that the oscillation period of correlations is only sightly shifted by a finite D. The total decay time, however, depends sensitively on the external noise level. For $D = 16$ and $D = 100$ the correlations $\tilde{C}_{zz}(t)$ and $\tilde{C}_{yy}(t)$ displayed in Fig. 5 for our two representative Rayleigh numbers below and above R_T are strikingly similar even without optimizing the time scale or the choice of parameters. This first of all means that the statistical dynamics of the y and z modes in the convective and the turbulent regimes is the same in the presence of certain noise levels if times are scaled appropriately. Since the correlations in y for $R = 30$ with and without external forcing do not differ considerably, we furthermore conclude from Fig. 5 the following: the statistical dynamics of the y degree of freedom in the unforced turbulent system is similar to the dynamics below R_T when external forces drive the system. This conclusion is supported by an inspection of the trajectories which display erratic behavior for $R < R_T$ and $D > 0$: a frequently spiraling type of motion back and forth around the two attracting points is observed—a feature characteristic of the trajectories in the unforced system above the turbulence threshold R_T. Despite all these similarities, we are somewhat reluctant to call this phenomenon a noise-induced transition to turbulence. The possible equivalence of external and internal noise which one is tempted to deduce from Fig. 5 should be investigated further.

4. SUMMARY

In this paper we have investigated the statistical properties of the Lorenz model for the Bénard problem driven by external random forces. Numerical calculations for three characteristic Rayleigh numbers and varying forcing strength were performed to analyze static and dynamic correlations in different flow states, with special emphasis on their dependence on the external noise level.

In the conductive regime $R < 1$, we compared our numerical results to the predictions of the linearized theory to estimate the importance of nonlinear terms as a function of external forcing strength D. We found that for increasing D the nonlinear terms grow in importance, producing erratic motion with phase mixing properties much as in the turbulent regime $R > R_T$ of the unforced system. This effect is observed "experimentally" as an increase in relaxation rates as compared to the linearized theory.

In the convective regime $1 < R < R_T$, the time-dependent correlations display damped oscillations, whose period we associate with the spiraling motion of trajectories around one of the attractors of the unforced system. The total decay rate is determined by the frequency of motion from one attractor to the other one. The oscillations are superimposed on an overall monotonic decay of correlations with the initial value, which is mainly caused by crossings of the trajectory between the two basins of attraction $x \gtrless 0$. The crossing frequency increases with increasing noise level D, and with it the decay rate.

Gaussian random forces restore the symmetry which is broken for $D = 0$ and $1 < R < R_T$. We found $\langle x \rangle = 0$, reflecting the trajectory's commuting between phase space regions with positive and negative x. In the limit $D \to 0$ the trajectory stays longer and longer near one attractor: We observe a "slowing down" effect in the dynamic correlations of x accompanying the transition to the state of broken symmetry for $D = 0$.

In the turbulent regime, $R > R_T$, our results show only small changes compared with the statistical behavior of the unforced system. The characteristic structure of the frequency spectrum of the autocorrelation functions is preserved in the presence of Gaussian forces with variance small compared to the average velocity squares. The external noise causes, however, an additional loss of memory, which is visible in the increased decay of correlations at larger times. A study for uniformly distributed external forces revealed the most gratifying fact that the dynamics of the system in the turbulent range is insensitive to the statistics of external forces. We finally discussed striking similarities in the statistical dynamics of the forced system in the convective regime and the unforced system in the turbulent regime.

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